DSP Lec 5

Quiz

$x(n) = \{1, 1, 1, 1\}$, $h(n) = \{2, 2\}$

Find $y(n) = x(n) * h(n)$
using inverse z-transform.

Sol

$$x(n) = \delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3)$$

$$h(n) = 2\delta(n) + 2\delta(n-1)$$

$$Y(z) = X(z) * H(z)$$

$$= \left[1 + z^{-1} + z^{-2} + z^{-3}\right]\left[2 + 2z^{-1}\right]$$

$$Y(z) = 2 + 4z^{-1} + 4z^{-2} + 4z^{-3} + 2z^{-4}$$

$$\boxed{y(n) = 2\delta(n) + 4\delta(n-1) + 4\delta(n-2) + 4\delta(n-3) + 2\delta(n-4)}$$

* Discrete Fourier Transform (DFT)

⇒ if we want to analysis for any signal, we want to evaluate some parameters as:

1. freq. content.
2. Power and energy density.
3. Amplitude.
4. Periodicity.

} → in frequency domain

← في بعض الـ APPs بنحتاج الحاجات دي.

~~مثلاً لما عايز نعمل transmition~~

useful for

↳ we want to convert signals from time domain to frequency domain to — study some parameters of the signals as (frequency content and power and energy density), which couldn't be studied in time domain.

* Some Applications depend on analysis of the signals on frequency domain as filter design.

$$x(t) \xrightarrow{\text{Laplace}} X(s)$$

time domain $\quad\quad\quad\quad$ s-domain

$$X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt$$

→ Continous Fourier transform (FT)

— إحنا مش بندرس الـ (Continous) لكنه دورنا نعرف إزاي حولنا من Cont. لـ (discrete)

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-J\omega t}\, dt} \longrightarrow \text{Continous Fourier transform}$$

— مشكلته :- إنك متقدرش تستخدمه مع الـ (digital systems)

— عشان كده لازم نقطع الـ (signal) دي ونحولها لـ (discrete)

$$X(\omega) \xrightarrow{\text{Discritization}} X(K)$$

ـــ تلخيص ما سبق :-

$$x(t) \xrightarrow{FT} X(\omega)$$

$$x(n) \xrightarrow{DFT} X(K)$$

ـــ لو ال (signal) اللي بتحولها (Periodic)

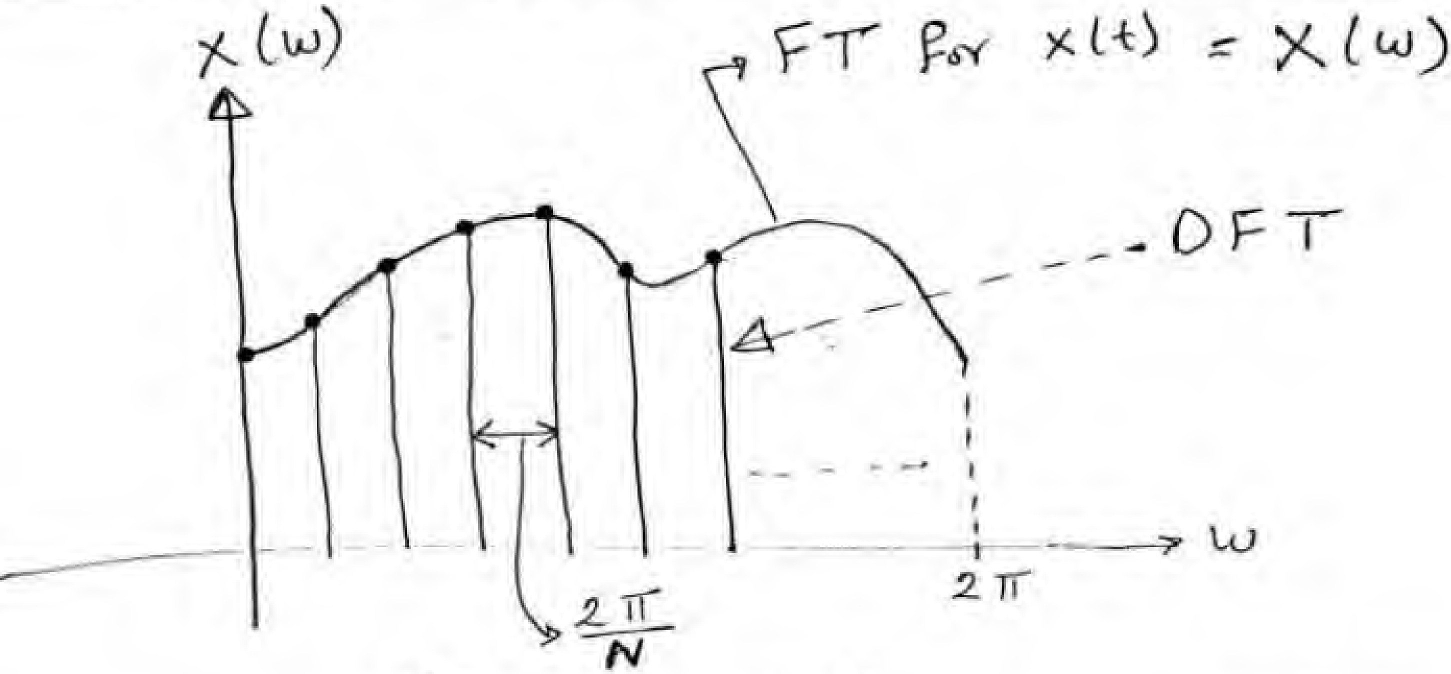
في ال (time domain) ف (frequency domain) هتنتج (Periodic signal) أيضاً.

$$\underset{\text{Periodic}}{\underline{x(t)}} \xrightarrow{FT} \underset{\text{Periodic}}{\underline{X(\omega)}}$$

and it's periodic every 2π interval of freq.

← بتتكرر كل 2π

* if, we assume the signals we deal with are periodic



$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \Rightarrow \text{F.T}$$

$\Rightarrow$ assume $X(\omega)$ is discretized to

N samples

$$\underbrace{\omega}_{\text{Continous Freq.}} \longrightarrow \frac{2\pi}{N} K$$

$K \rightarrow$ the sample number for DFT.

$(0, 1, ----, N-1)$

$N \Rightarrow$ the no. of samples for D.F.T.

$n = 0, 1, 2, \ldots, N-1$

$K = 0, 1, 2, \ldots, N-1$

D.F.T

$$\boxed{X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-J \frac{2\pi}{N} K n}}$$

$n \Rightarrow$ the sample no. for discrete time sequence $x(n)$, $n = 0, 1, 2, \ldots, N-1$

– هل لازم ال no. of sampling في X(K) يكون هو نفسه ال(no. of sampling) لـ x(n)

– لأ مش لازم.

– لكن هنفترض إنه (no. of sampling) واحد.

– بس ~~خلي~~ يكون (no. of sampling) لـ X(K) أكبر من ال (" " ") لـ x(n)

– بس لو عندنا no. of sampling = 4 لـ x(n) و no. of sampling = 10 لـ X(K)

– هنحط الباقي أصفار ~~ـ~~ عشان كده بنفترض إنهم متساويين.

$\boxed{Ex}$ $x(n) = \{0,1,2,3\}$, find the 4-point discrete fourier transform of $x(n)$ ⇒ given $N = 4$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-J\frac{2\pi}{N}Kn}$$

<u>$K = 0$</u>

$$X(0) = \sum_{n=0}^{3} x(n) = x(0) + x(1) + x(2) + x(3)$$

$$= 6$$

<u>$K = 1$</u>

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{-J\frac{2\pi}{4}n}$$

$$= x(0) + x(1)\, e^{-J\frac{\pi}{2}} + x(2)\, e^{-J\pi} + x(3)\, e^{-J\frac{3\pi}{2}}$$

$$= e^{-J\frac{\pi}{2}} + 2\, e^{-J\pi} + 3\, e^{-J\frac{3\pi}{2}}$$

<u>Note that</u>

$$e^{-J\theta} = \cos\theta - J\sin\theta$$

$$X(1) = \cos(90) - J\sin(90) + 2\left(\cos(180) - J\sin(180)\right)$$

$$+3\left(\cos(270) - J\sin(270)\right)$$

$$X(1) = -J - 2 + 3J = -2 + 2J$$

* $\underline{\underline{K=2}}$

$$X(2) = \sum_{n=0}^{3} x(n)\, e^{-J\frac{4\pi}{4}n} = \sum_{n=0}^{3} x(n)\, e^{-J\pi n}$$

$$= \underset{\downarrow\, 0}{x(0)} + \underset{\downarrow\, 1}{x(1)}\, e^{-J\pi} + \underset{\downarrow\, 2}{x(2)}\, e^{-J2\pi} + \underset{\downarrow\, 3}{x(3)}\, e^{-J3\pi}$$

$$= e^{-J\pi} + 2e^{-J2\pi} + 3e^{-J3\pi}$$

$$= \left(\cos(180) - J\sin(180)\right) + 2\left(\cos(360) - J\sin(360)\right)$$

$$+ 3\left(\cos(180) - J\sin(180)\right)$$

$$X(2) = -1 + 2 - 3 = -2$$

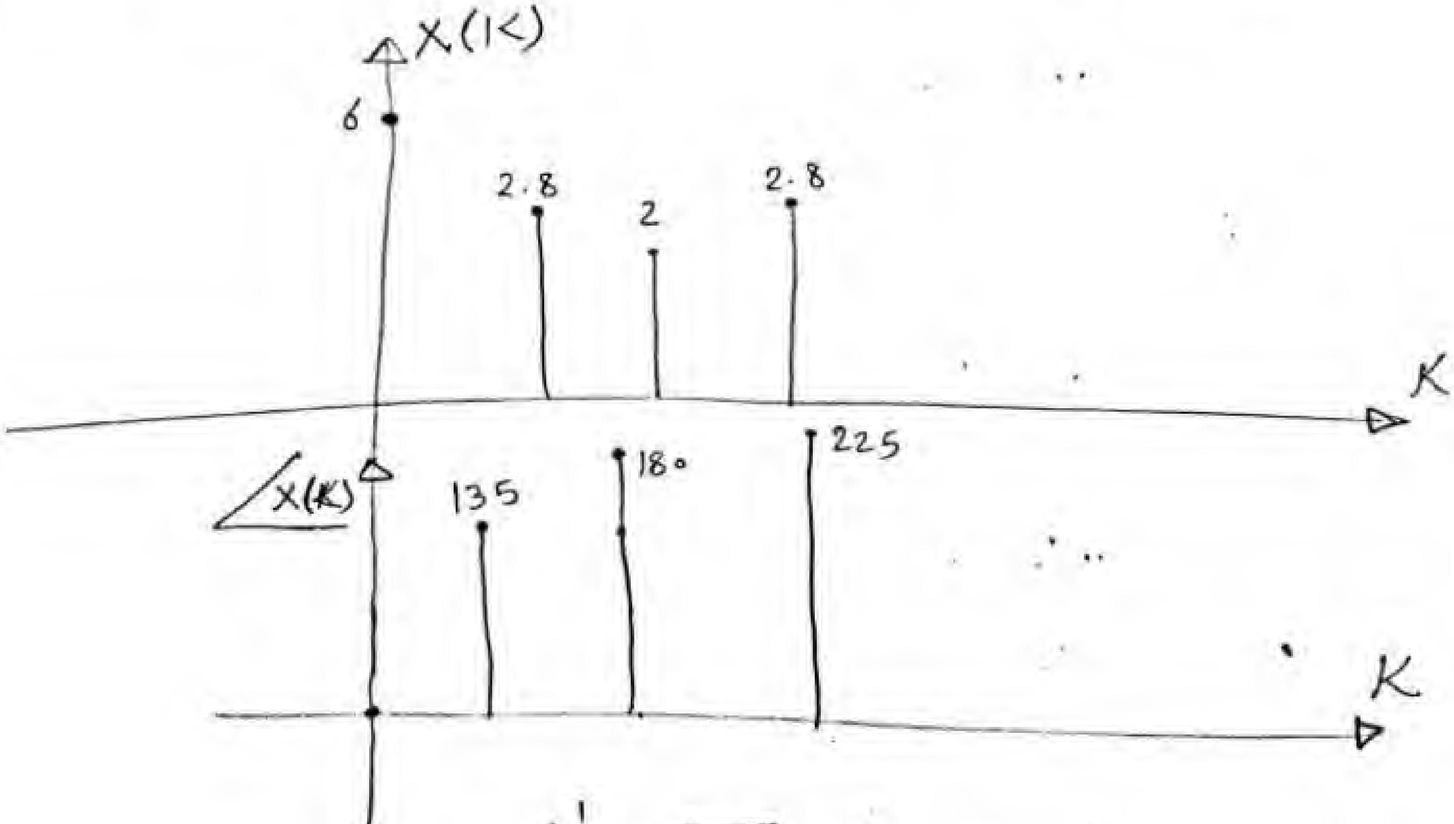
$\underline{\underline{K = 3}}$

$$X(3) = -2 - 2J$$

$$\underset{\substack{\downarrow \\ \text{DFT for} \\ x(n)}}{\underline{X(K)}} = \{ 6, -2 + J2, -2, -2 - 2J \}$$

$$|X(K)| = \{ 6, 2\sqrt{2}, 2, 2\sqrt{2} \}$$

$$\angle X(K) = \{ 0, 135, 180, \underbrace{225}_{\text{or } -135} \}$$



– ف الإمتحان هيجيب $N=3$ or $N=4$

– صورة الحل الماضية لو حليت بيها عادى لكن القادم حل أبسط ولك حرية الإختيار في الحل.

$$x(n) \xrightarrow{DFT} X(K) = \sum_{n=0}^{N-1} x(n) \, e^{-J\frac{2\pi}{N}nK}$$

$$W_N = e^{-J\frac{2\pi}{N}}$$

$$X(K) = \sum_{n=0}^{N-1} x(n) \, (W_N)^{Kn}$$

$K = 0, 1, 2, \ldots, N-1$

$n = 0, 1, 2, \ldots, N-1$

$$X_N = \begin{bmatrix} X(K=0) \\ X(K=1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$x_N = \begin{bmatrix} x(n=0) \\ x(n=1) \\ \vdots \\ x(n=N-1) \end{bmatrix}$$

$$X_N = \begin{array}{c|c} & n=0 \quad \cdots \quad n=N-1 \\ \begin{array}{c} k=0 \\ \vdots \\ k=N-1 \end{array} & \left[W_N^{kn} \right]_{N*N} \end{array} \cancel{x} * x_N$$

<u>For example</u> $N = 4$

$$W_4 = \begin{array}{c|c} & \begin{array}{cccc} n=0 & n=1 & n=2 & n=3 \end{array} \\ \begin{array}{c} k=0 \\ k=1 \\ k=2 \\ k=3 \end{array} & \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^1 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \end{array}$$

$$W_4 = e^{-J\frac{2\pi}{4}} = \cos(90) - J\sin(90) = -J$$

$$W_4^0 = 1 \quad , \quad W_4^2 = e^{-J\pi} = \cos(180) - J\sin(180) = -1$$

$$W_4^3 = e^{-J\frac{3\pi}{2}} = \cos(270) - J\sin(270) = J$$

$$W_4^4 = W_4^0 = 1 \quad , \quad W_4^9 = W_4^1 = -J$$

$$W_4^6 = W_4^2 = -1$$

$$W_N^N = e^{-J\frac{2\pi}{N}N} = e^{-J2\pi} = 1$$

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -J & -1 & J \\ 1 & -1 & 1 & -1 \\ 1 & J & -1 & -J \end{bmatrix}$$

another solution

DFT for $x(n)$ $(N=4)$

$$X_4 = [W_4]\, x_4$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -J & -1 & J \\ 1 & -1 & 1 & -1 \\ 1 & J & -1 & -J \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$X_4 = \begin{bmatrix} 6 \\ -2+2J \\ -2 \\ -2-2J \end{bmatrix}$$

For $N=3$ $\quad [W_3] = \left[W_3^{Kn} \right]_{3*3}$

$$= \begin{array}{c} \\ K=0 \\ K=1 \\ K=2 \end{array} \begin{array}{c} \begin{array}{ccc} n=0 & n=1 & n=2 \end{array} \\ \begin{bmatrix} W_3^0 & W_3^0 & W_3^0 \\ W_3^0 & W_3^1 & W_3^2 \\ W_3^0 & W_3^2 & W_3^4 \end{bmatrix} \end{array}$$

where $W_3 = e^{-J\frac{2\pi}{3}}$

$* \; W_3^0 = 1$

$* \; W_3^1 = e^{-J\frac{2\pi}{3}} = \cos(120) - J\sin(120)$

$= -0.5 - J\frac{\sqrt{3}}{2}$

$* \; W_3^2 = e^{-J\frac{4\pi}{3}} = \cos(240) - J\sin(240)$

$= -0.5 + J\frac{\sqrt{3}}{2}$

$* \; W_3^4 = W_3^1 = -0.5 - J\frac{\sqrt{3}}{2}$;

$$[W_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -0.5 - J\frac{\sqrt{3}}{2} & -0.5 + J\frac{\sqrt{3}}{2} \\ 1 & -0.5 + J\frac{\sqrt{3}}{2} & -0.5 - J\frac{\sqrt{3}}{2} \end{bmatrix}$$

$\boxed{Ex}$ for $x(n) = \{0.5, 1\}$ Determine 3-Point DFT.

For $N = 3$ $\quad X_3 = [W_3] \; x_3$; $x_3 = \begin{bmatrix} 0.5 \\ 1 \\ 0 \end{bmatrix}$

$$X_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -0.5 - J\frac{\sqrt{3}}{2} & -0.5 + J\frac{\sqrt{3}}{2} \\ 1 & -0.5 + J\frac{\sqrt{3}}{2} & -0.5 - J\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \\ 0 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 1.5 \\ -J\frac{\sqrt{3}}{2} \\ J\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$X(K) = \left\{ 1.5, -j\frac{\sqrt{3}}{2}, j\frac{\sqrt{3}}{2} \right\}$$

$$|X(K)| = \left\{ 1.5, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \right\}$$

$$\angle X(K) = \left\{ 0, -90^\circ, 90^\circ \right\}$$